B.Sc Part - I

RELATIVISTIC LAGRANGIAN

From classical mechanics we know the expression of Lagrangian

$$L = T - V$$

where T = K.E of system and it is function of generalised moment P_r of generalized velocities q_r.

V = P.E of system and function of generalised co-ordinates q_r only.

Now as studied earlier, the relativistic K.E of a particle of rest mass m_0 moving with velocity $u = \alpha c$ is

$$T = m_0 c^2 \left[\frac{1}{\sqrt{(1-\alpha^2)}} - 1 \right] \quad \text{--- (1)}$$

and relativistic momentum is given by

$$P = mv = \frac{m_0 u}{\sqrt{(1-\alpha^2)}} \quad \text{--- (2)}$$

we know by definition

$$\left.\begin{aligned} \frac{\partial T}{\partial x} &= P_x \\ \frac{\partial T}{\partial y} &= P_y \\ \frac{\partial T}{\partial z} &= P_z \end{aligned}\right\} \quad \text{--- (3)}$$

we cannot get the relativistic compound of momentum as given by eqn (1) by differentiation eq (1)

we therefor, define a function T^* such that

$$P_x = \frac{m_0 \dot{x}}{\sqrt{(1-\alpha^2)}} = \frac{\partial T^*}{\partial \dot{x}}$$

$$P_y = \frac{m_0 \dot{y}}{\sqrt{(1-\alpha^2)}} = \frac{\partial T^*}{\partial \dot{y}} \quad \longrightarrow (4)$$

$$P_z = \frac{m_0 \dot{z}}{\sqrt{(1-\alpha^2)}} = \frac{\partial T^*}{\partial \dot{z}}$$

Now K.E T^* is a function of velocity components $\dot{x}, \dot{y}, \dot{z}$ we have

$$T^* = T^*(\dot{x}, \dot{y}, \dot{z})$$

$$dT^* = \frac{\partial T^*}{\partial \dot{x}} d\dot{x} + \frac{\partial T^*}{\partial \dot{y}} d\dot{y} + \frac{\partial T^*}{\partial \dot{z}} d\dot{z}$$

we get from eqn (4)

$$dT^* = \frac{m_0}{\sqrt{(1-\alpha^2)}} (\dot{x} d\dot{x} + \dot{y} d\dot{y} + \dot{z} d\dot{z})$$

$$= \frac{m_0}{\sqrt{(1-\alpha^2)}} u\,du \qquad \left\{ \begin{array}{l} \text{as } u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ \text{or } u\,du = \dot{x} d\dot{x} + \dot{y} d\dot{y} + \dot{z} d\dot{z} \end{array} \right.$$

$$= \frac{m_0 (c\alpha)\, c\, d\alpha}{(1-\alpha^2)^{1/2}} \qquad \left[\begin{array}{l} \text{as } u = c\alpha \\ \text{or } du = c\,d\alpha \end{array} \right.$$

$$= \frac{m_0 c^2 \alpha\, d\alpha}{\sqrt{(1-\alpha^2)}}$$

Integrating above we get

$$T^* = -m_0c^2\sqrt{(1-\alpha^2)} + A$$

To get A when u is much smaller in comparison with c, $T^* = \frac{1}{2}m_0u^2$, the previous equation gives

$$\frac{1}{2}m_0u^2 = -m_0c^2 + A$$

$\left[\because \alpha = u/c \to 0 \text{ as } u << c\right]$

$$A = \frac{1}{2}m_0u^2 + m_0c^2$$

$$A = m_0c^2$$

Hence we get

$$T^* = -m_0c^2\sqrt{(1-\alpha^2)} + m_0c^2$$

$$T^* = m_0c^2\left[1-\sqrt{(1-\alpha^2)}\right] \quad \text{——} \quad (5)$$

So Relativistic Lagrangian function is given by

$$L = T^* - V \quad \text{——} \quad (6)$$

V is independent of velocity hence it remains unchanged in relativistic.

Equation of motion
we know from classical mechanics

$$P_r = \frac{\partial L}{\partial \dot{q}_r} = \frac{\partial}{\partial \dot{q}_r}(T-V) = \frac{\partial T}{\partial \dot{q}_r}$$

As $\frac{\partial V}{\partial \dot{q}_r} = 0$ since V does not depend on $\dot{q}_r$ we get

$$P_r = \frac{\partial T}{\partial \dot{q}_r} = \frac{\partial L}{\partial \dot{q}_r} \quad \text{— (7)}$$

$$\frac{dP_r}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)$$

$$\frac{dP_r}{dt} = \frac{\partial L}{\partial q_r} \quad \text{— (8)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

(From Lagrangian eqn
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \text{)}$$

The eqn (7) is classical eqn. similar to it in relatively the eqn is

$$\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T^*}{\partial \dot{q}_r}$$

and from (7) we get

$$\frac{\partial L}{\partial q_r} = -\frac{\partial V}{\partial q_r} = -\frac{\partial P_r}{\partial t}$$

($\because L = T - V$ as V is a function of q_r only)

$$\frac{d}{dt}\left[\frac{m_0 \dot{x}}{\sqrt{(1-\alpha^2)}}\right] = -\frac{\partial V}{\partial x}$$

$$\frac{d}{dt}\left[\frac{m_0 \dot{y}}{\sqrt{(1-\alpha^2)}}\right] = -\frac{\partial V}{\partial y}$$

$$\frac{d}{dt}\left[\frac{m_0 \dot{z}}{\sqrt{(1-\alpha^2)}}\right] = -\frac{\partial V}{\partial z}$$

These are relativistic equation of motion

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